Monday, October 26, 2015

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Problem 19

Problem. Find the indefinite integral $\int \sec 4x \, dx$.

Solution.

$$\int \sec 4x \, dx = \frac{1}{4} \ln|\tan 4x + \sec 4x| + C.$$

Problem 21

Problem. Find the indefinite integral $\int \sec^3 \pi x \, dx$.

Solution. First, clean it up a bit with the substitution $u = \pi x$, $du = \pi \, dx$ just to be safe. We get

$$\int \sec^3 \pi x \, dx = \frac{1}{\pi} \int \sec^3 u \, du.$$

Now use the formula that we worked out in class.

$$\begin{aligned} \frac{1}{\pi} \int \sec^3 u \, du &= \frac{1}{2\pi} (\tan u \sec u + \ln|\sec u + \tan u|) + C \\ &= \frac{1}{2\pi} (\tan \pi x \sec \pi x + \ln|\sec \pi x + \tan \pi x|) + C. \end{aligned}$$

Problem 24

Problem. Find the indefinite integral $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx$.

Solution. Let's clean this one up, too, with $u = \frac{\pi x}{2}$, $du = \frac{\pi}{2} \, dx$. We get

$$\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2} \, dx = \frac{2}{\pi} \int \tan^3 u \sec^2 u \, du.$$

Now use our technique.

$$\frac{2}{\pi} \int \tan^3 u \sec^2 u \, du = \frac{2}{\pi} \int \tan u(\sec^2 u - 1) \sec^2 u \, du.$$

Let $v = \sec u$ and $dv = \tan u \sec u \; du$. Then

$$\begin{aligned}
\frac{2}{\pi} \int \tan u(\sec^2 u - 1) \sec^2 u \; du &= \frac{2}{\pi} \int (\sec^2 u - 1) \sec u (\tan u \sec u) \; du \\
&= \frac{2}{\pi} \int (v^2 - 1) v \; dv \\
&= \frac{2}{\pi} \int (v^3 - v) \; dv \\
&= \frac{2}{\pi} \left(\frac{1}{4} v^4 - \frac{1}{2} v^2 \right) + C \\
&= \frac{2}{\pi} \left(\frac{1}{4} \sec^4 u - \frac{1}{2} \sec^2 u \right) + C \\
&= \frac{2}{\pi} \left(\frac{1}{4} \sec^4 \frac{\pi x}{2} - \frac{1}{2} \sec^2 \frac{\pi x}{2} \right) + C \\
&= \frac{1}{2\pi} \sec^4 \frac{\pi x}{2} - \frac{1}{\pi} \sec^2 \frac{\pi x}{2} + C.
\end{aligned}$$

Problem 25

Problem. Find the indefinite integral $\int \tan^3 2t \sec^3 2t \; dt$.

Solution. Let $u = 2t$, $du = 2 \; dt$. Then

$$\begin{aligned}
\int \tan^3 2t \sec^3 2t \; dt &= \frac{1}{2} \int \tan^3 u \sec^3 u \; du \\
&= \frac{1}{2} \int \tan u(\sec^2 u - 1) \sec^3 u \; du.
\end{aligned}$$

Let $v = \sec u$ and $dv = \tan u \sec u\ du$. Then

$$\begin{aligned}
\frac{1}{2}\int \tan u(\sec^2 u - 1)\sec^3 u\ du &= \frac{1}{2}\int (\sec^2 u - 1)\sec^2 u(\tan u \sec u)\ du \\
&= \frac{1}{2}\int (v^2 - 1)v^2\ dv \\
&= \frac{1}{2}\int (v^4 - v^2)\ dv \\
&= \frac{1}{2}\left(\frac{1}{5}v^5 - \frac{1}{3}v^3\right) + C \\
&= \frac{1}{10}v^5 - \frac{1}{6}v^3 + C \\
&= \frac{1}{10}\sec^5 u - \frac{1}{6}\sec^3 u + C \\
&= \frac{1}{10}\sec^5 2t - \frac{1}{6}\sec^3 2t + C
\end{aligned}$$

Problem 26

Problem. Find the indefinite integral $\int \tan^5 2x \sec^4 2x\ dx$.

Solution. Let $u = 2x$, $du = 2\ dx$. Then

$$\begin{aligned}
\int \tan^5 2x \sec^4 2x\ dt &= \frac{1}{2}\int \tan^5 u \sec^4 u\ du \\
&= \frac{1}{2}\int \tan u(\sec^2 u - 1)^2 \sec^4 u\ du.
\end{aligned}$$

Let $v = \sec u$ and $dv = \tan u \sec u \, du$. Then

$$\begin{aligned}
\frac{1}{2}\int \tan u(\sec^2 u - 1)^2 \sec^4 u \, du &= \frac{1}{2}\int (\sec^2 u - 1)^2 \sec^3 u(\tan u \sec u) \, du \\
&= \frac{1}{2}\int (v^2 - 1)^2 v^3 \, dv \\
&= \frac{1}{2}\int (v^7 - 2v^5 + v^3) \, dv \\
&= \frac{1}{2}\left(\frac{1}{8}v^8 - \frac{1}{3}v^6 + \frac{1}{4}v^4\right) + C \\
&= \frac{1}{16}v^8 - \frac{1}{6}v^6 + \frac{1}{8}v^4 + C \\
&= \frac{1}{16}\sec^8 u - \frac{1}{6}\sec^6 u + \frac{1}{8}\sec^4 u + C \\
&= \frac{1}{16}\sec^8 2x - \frac{1}{6}\sec^6 2x + \frac{1}{8}\sec^4 2x + C.
\end{aligned}$$

Problem 30

Problem. Find the indefinite integral $\int \tan^3 3x \, dx$.

Solution.

$$\begin{aligned}
\int \tan^3 3x \, dx &= \int \tan 3x(\sec^2 3x - 1) \, dx \\
&= \int \tan 3x \sec^2 3x \, dx - \int \tan 3x \, dx
\end{aligned}$$

For the first integral, let $u = \sec 3x$ and $du = 3\tan 3x \sec 3x \, dx$. The second integral we can integrate directly.

$$\begin{aligned}
\int \tan 3x \sec^2 3x \, dx - \int \tan 3x \, dx &= \frac{1}{3}\int \sec 3x(3\tan 3x \sec 3x) \, dx - \frac{1}{3}\ln|\sec 3x| + C \\
&= \frac{1}{3}\int u \, du - \frac{1}{3}\ln|\sec 3x| + C \\
&= \frac{1}{3}\left(\frac{1}{2}u^2\right) - \frac{1}{3}\ln|\sec 3x| + C \\
&= \frac{1}{6}u^2 - \frac{1}{3}\ln|\sec 3x| + C \\
&= \frac{1}{6}\sec^2 3x - \frac{1}{3}\ln|\sec 3x| + C.
\end{aligned}$$

Problem 42

Problem. Find the indefinite integral $\int \cos 5\theta \cos 3\theta \, d\theta$.

Solution. Use the identity

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)).$$

$$\begin{aligned}\int \cos 5\theta \cos 3\theta \, d\theta &= \frac{1}{2}\int (\cos 2\theta + \cos 8\theta)\, d\theta \\ &= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \frac{1}{8}\sin 8\theta\right) + C \\ &= \frac{1}{4}\sin 2\theta + \frac{1}{16}\sin 8\theta + C.\end{aligned}$$

Problem 43

Problem. Find the indefinite integral $\int \sin 2x \cos 4x \, dx$.

Solution. Use the identity

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)).$$

$$\begin{aligned}\int \sin 2x \cos 4x \, dx &= \frac{1}{2}\int (\sin(-2x) + \sin 6x)\, dx \\ &= \frac{1}{2}\int (-\sin 2x + \sin 6x)\, dx \\ &= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{6}\cos 6x\right) + C \\ &= \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C.\end{aligned}$$

Problem 46

Problem. Find the indefinite integral $\int \sin 5x \sin 4x \, dx$.

Solution. Use the identity

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)).$$

$$\begin{aligned}\int \sin 5x \sin 4x\ dx &= \frac{1}{2}\int (\cos x - \cos 9x)\ dx \\ &= \frac{1}{2}\left(\sin x - \frac{1}{9}\sin 9x\right) + C \\ &= \frac{1}{2}\sin x - \frac{1}{18}\sin 9x + C.\end{aligned}$$

Problem 59

Problem. Evaluate the definite integral $\displaystyle\int_0^{\pi/4} 6\tan^3 x\ dx$.

Solution. Based on Exercise 30, we know that

$$\int \tan^3 x\ dx = \frac{1}{2}\sec^2 x - \ln|\sec x| + C.$$

Now we can evaluate the definite integral.

$$\begin{aligned}\int_0^{\pi/4} 6\tan^3 x\ dx &= 6\left[\frac{1}{2}\sec^2 x - \ln|\sec x|\right]_0^{\pi/4} \\ &= \big[3\sec^2 x - 6\ln|\sec x|\big]_0^{\pi/4} \\ &= \left(3\sec^2\frac{\pi}{4} - 6\ln|\sec\frac{\pi}{4}|\right) - \big(3\sec^2 0 - 6\ln|\sec 0|\big) \\ &= \left(3(\sqrt{2})^2 - 6\ln|\sqrt{2}|\right) - \big(3(1)^2 - 6\ln|1|\big) \\ &= 6 - 3\ln 2 - 3 \\ &= 3 - 3\ln 2.\end{aligned}$$

Problem 60

Problem. Evaluate the definite integral $\displaystyle\int_0^{\pi/3} \sec^{3/2} x\tan x\ dx$.

Solution. Let $u = \sec x$ and $du = \tan x \sec x \, dx$. Then

$$\begin{aligned}
\int_0^{\pi/3} \sec^{3/2} x \tan x \, dx &= \int_0^{\pi/3} \sec^{1/2} x (\tan x \sec x) \, dx \\
&= \int_1^2 u^{1/2} \, du \\
&= \left[\frac{2}{3} u^{3/2}\right]_1^2 \\
&= \frac{2}{3}\left(2^{3/2} - 1\right) \\
&= \frac{4\sqrt{2} - 2}{3}.
\end{aligned}$$

Problem 63

Problem. Evaluate the definite integral $\displaystyle\int_{-\pi/2}^{\pi/2} 3\cos^3 x \, dx$.

Solution.

$$\int_{-\pi/2}^{\pi/2} 3\cos^3 x \, dx = 3\int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) \cos x \, dx.$$

Let $u = \sin x$ and $du = \cos x \, dx$. Then

$$\begin{aligned}
3\int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) \cos x \, dx &= 3\int_{-1}^{1} (1 - u^2) \, du \\
&= 3\left[u - \frac{1}{3}u^3\right]_{-1}^{1} \\
&= 3\left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right) \\
&= 4.
\end{aligned}$$